RADIANT HEATING OF SIMPLE SHAPES

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Steady and unsteady radiant heating of a plate and a cylinder are analyzed, and a first-approximation solution is obtained by a method of averaging correction functions.

Consider symmetrical heating of a plate of thickness 2b and a cylinder of radius R, the end faces of which are thermally insulated. It is assumed that the heating conditions of both plate and cylinder are those corresponding to a muffle furnace. Then, according to [1], about 90% of the heating of the metal in the furnace space is due to radiative heat transfer.

In what follows, therefore, the convection component will be neglected, and the assumption made that only radiant heating occurs.

Mathematically, the problem is that of solving the equation of thermal conduction with corresponding boundary conditions:

for the plate:

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}, \tag{1}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=b} = h\,[T_c^4 - T_s^4], \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0; \tag{3}$$

for the cylinder:

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{1'}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = h\,[T_c^4 - T_п^4], \tag{2'}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0. \tag{3'}$$

Conditions (2) and (2') reflect the well-known Stefan-Boltzmann law, while conditions (3) and (3') are based on symmetry of heating.

Let us assume that the temperature of the heating medium T_C and all the thermophysical parameters (a; $h = \sigma_V/\lambda$) are constant.

For simplicity, we assume uniform initial conditions

$$T(x,\ 0) = T_0 = 0; \tag{4}$$

$$T(r,\ 0) = T_0 = 0. \tag{4'}$$

Introducing new dimensionless variables and a relative temperature function

$$\xi = hT_c^3 x,\ \ \tau = ah^2T_c^6 t,\ \ u(\xi,\ \tau) = \frac{T(\xi,\ \tau)}{T_c}, \tag{5}$$

$$\rho = hT_c^3 r,\ \ \tau = ah^2T_c^6 t,\ \ u(\rho,\ \tau) = \frac{T(\rho,\ \tau)}{T_c}, \tag{5'}$$

We transform equations (1), (1'), and conditions (2), (2'), (3), (3'), (4), and (4') as follows:

for the plate:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2}, \tag{6}$$

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=\beta} = (1 - u_s^4), \tag{7}$$

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=0} = 0, \tag{8}$$

$$u(\xi, 0) = u_0 = 0, \tag{9}$$

where

$$\beta = hT_c^3 b; \tag{10}$$

for the cylinder:

$$\frac{\partial u}{\partial \rho} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}, \tag{6'}$$

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=\delta} = (1 - u_s^4), \tag{7'}$$

$$\left.\frac{\partial u}{\partial \rho}\right|_{\rho=0} = 0. \tag{8'}$$

$$u(\rho, 0) = u_0 = 0, \tag{9'}$$

where

$$\delta = hT_c^3 R. \tag{10'}$$

Here we have adopted a rather unusual method of substituting variables (cf. [2]) in order to write (1), (1'), (2) and (2') free of coefficients.

To solve the problem, two stages must be distinguished – unsteady and steady heating.

Unsteady heating: To investigate the initial phase of heating, when the heat front has not yet reached the middle of the plate, but has penetrated only to a certain depth $\gamma(\tau)$, (6) must be solved for the somewhat different boundary conditions

$$\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=0} = -(1 - u_{1s}^4); \tag{11}$$

$$u_1(\xi, \tau)|_{\xi=\gamma(\tau)} = u_0 = 0; \tag{12}$$

$$\left.\frac{\partial u_1}{\partial \xi}\right|_{\xi=\gamma(\tau)} = 0. \tag{13}$$

Expressions (12) and (13) are the conditions of conjugation of the function $u_1(\xi,\tau)$ and the initial temperature $u_0 = 0$ at the boundary between the heated and unheated parts of the plate. It should be noted that the origin of coordinates is assumed to be not on the axis of symmetry, but at the left edge of the plate, in contrast to that for (6)-(9).

For an approximate solution of this problem, the method of averaging correction functions [3, 4] is used, according to which, in the first approximation we put

$$\frac{\partial^2 u_1}{\partial \xi^2} = f_1(\tau), \tag{14}$$

where $f_1(\tau)$ is the average rate of increase in temperature with respect to ξ in the region of perturbation $\gamma(\tau)$.

$$f_1(\tau) = \frac{1}{\gamma(\tau)} \int_0^{\gamma(\tau)} \frac{\partial u_1}{\partial \tau} d\xi. \tag{15}$$

Integrating (14) twice, we have

$$u_1(\xi, \tau) = f_1(\tau)\frac{\xi^2}{2} + A_1(\tau)\xi + B_1(\tau). \tag{16}$$

Using boundary conditions (12), (13), we may express the unknown functions of integration $A_1(\tau)$ and $B_1(\tau)$ in terms of $f_1(\tau)$ and $\gamma(\tau)$:

$$A_1(\tau) = -f_1(\tau)\gamma(\tau), \tag{17}$$

$$B_1(\tau) = \frac{f_1(\tau)\gamma^2(\tau)}{2}. \tag{18}$$

Substituting (17) and (18) in (16), we find

$$u_1(\xi, \tau) = u_{u_{1s}}\left[1 - \frac{\xi}{\gamma(\tau)}\right]^2, \tag{19}$$

where

$$u_{u_{1s}} = u_1(0, \tau) = \frac{f_1(\tau)\gamma^2(\tau)}{2}. \tag{20}$$

The connection between functions $\gamma(\tau)$ and $f_1(\tau)$ can be expressed by the nonlinear relation

$$\frac{f_1(\tau)\gamma^2(\tau)}{2} = \sqrt[4]{1 - f_1(\tau)\gamma(\tau)}, \tag{21}$$

which is obtained from condition (11), using (16), (17) and (18).

Introducing the new function

$$v_1(\tau) = f_1(\tau)\gamma(\tau), \tag{22}$$

we have, from (21)

$$\frac{v_1(\tau)\gamma(\tau)}{2} = \sqrt[4]{1 - v_1(\tau)}, \tag{23}$$

where

$$\gamma(\tau) = 2\frac{\sqrt[4]{1 - v_1(\tau)}}{v_1(\tau)}. \tag{24}$$

To determine function $v_1(\tau)$, we use condition (15), into which we introduce the derivative of (16) with respect to time τ, taking into account (17), (18), (22) and (24). After the appropriate mathematical transformations, we arrive at the following differential equation relating to the unknown function $v_1(\tau)$:

$$\frac{2 - v_1(\tau)}{v_1^3(\tau)\sqrt{1 - v_1(\tau)}}\, dv_1(\tau) = -3d\tau. \tag{25}$$

To (25) we must add the initial condition

$$v_1(0) = 1, \tag{26}$$

which follows from (24), since when $\tau = 0$ $\gamma(0) = 0$.

Integration of (25), taking into account (26), leads to the following expression:

$$\frac{1}{2}\ln\frac{1 - \sqrt{1 - v_1(\tau)}}{1 + \sqrt{1 - v_1(\tau)}} - \frac{2 + v_1(\tau)}{v_1^2(\tau)}\sqrt{1 - v_1(\tau)} = -6\tau. \tag{27}$$

It is not possible to obtain an exact expression for the function $v_1(\tau)$ from this transcendental equation, but this is not necessary, since an approximate solution will suffice. From (20) and (23) we may write

$$v_1(\tau) = 1 - u_{1s}^4(\tau). \tag{28}$$

After substituting (28) in (27) we have

$$\frac{1}{2}\ln\frac{1+u_{1s}^2}{1-u_{1s}^2} + \frac{3-u_{1s}^4}{(1-u_{1s}^4)^2}u_{1s}^2 = 6\tau. \tag{29}$$

Since the period of unsteady heating is usually short, the relative temperature of the surface $u_{1s} \ll 1$. In this case, expanding $\ln\frac{1+u_{1s}^2}{1-u_{1s}^2}$ in a power series, and neglecting terms containing the factor u_{1s}^k ($k \geq 4$) as compared with unity, we obtain from (29) the approximation

$$u_{1s}(\tau) = \frac{\sqrt{6\tau}}{2}, \tag{30}$$

which is valid correct to 5% for $u_{1s} \ll 0.625$.

Taking into account (28), (30) and (24), we obtain a formula defining the zone of thermal perturbation

$$\gamma(\tau) = \frac{\sqrt{6\tau}}{1-2.25\tau^2}. \tag{31}$$

Substituting (30) and (31) in (19), we finally arrive at the following approximate expression for the relative temperature function for unsteady radiant heating and zero initial conditions

$$u_1(\xi, \tau) = \frac{\sqrt{6\tau}}{2}\left[1 - \frac{(1-2.25\tau^2)}{\sqrt{6\tau}}\xi\right]^2. \tag{32}$$

By means of (5) we can return to the original notation in formulas (31) and (32)

$$T_1(x, t) = hT_c^4\frac{\sqrt{6at}}{2}\left(1 - \frac{1-2.25a^2h^4T_c^{12}t^2}{\sqrt{6at}}\right)^2; \tag{33}$$

$$l(t) = \frac{\sqrt{6at}}{1-2.25a^2h^4T_c^{12}t^2}. \tag{34}$$

The period of unsteady heating t_0 is usually small. The term $2.25a^2h^4T_c^{12}t^2$ in (33) and (34) can therefore be neglected. Then (33) and (34) take the simplified form:

$$T_1(x, t) = hT_c^4\frac{\sqrt{6at}}{2}\left(1 - \frac{x}{\sqrt{6at}}\right)^2; \tag{35}$$

$$l(t) = \sqrt{6at}.$$

The rate of propagation of the thermal perturbation may be obtained from (35)

$$w(t) = \frac{dl}{dt} = \sqrt{\frac{3a}{2t}}, \tag{36}$$

and also the period of unsteady heating of the plate

$$t_0 = \frac{b^2}{6a}. \tag{37}$$

Let us establish the limits of application of the simplified formulas (35)-(38). For this we shall take the following values of the thermophysical parameters of the problem: $T_c = 1200°K$; $\sigma_V = 3 \cdot 10^{-8} \cdot 1.1630$ w/m^2 $\cdot$ degree;

$\lambda = 30 \cdot 1.1630$ w/m · degree; $a = \frac{0.025}{3600}$ m²/sec.

For these values of the parameters, (35)-(38) may be used for $t_0 \leq 72 \cdot 10^2$ sec correct to 5%. From (38) we can also establish the limiting plate thickness for which the foregoing simplified formulas are valid:

$$2b \leqslant 2\sqrt{6at_0} = 1.1 \text{ m}. \tag{38}$$

If we introduce the well-known Fourier and Stark parameters

$$\mathrm{Fo}^0 = \frac{al_0}{b^2}\,;\; \mathrm{Sk} = hT_c^3 b, \tag{39}$$

then, in the range $0 \leq t_0 \leq 2$ hr, $0 \leq b \leq 0.55$ m, the criteria (39) assume the following values:

$$0 \leqslant \mathrm{Fo}^0 \leqslant 0.17 \;\; (b = 0.55 \text{ mm};\; 0 \leqslant t_0 \leqslant 2 \text{ hr});$$
$$\infty > \mathrm{Fo}^0 \geqslant 0.17 \;\; (t_0 = 2 \text{ hr.};\; 0 < b \leqslant 0.55 \text{ m});$$
$$0 \leqslant \mathrm{sk} \leqslant 1.0 \;\; (0 \leqslant b \leqslant 0.55 \text{ m}).$$

In calculating the heating of thicker plates, (33) and (34) must be used.

Using the above method [3, 4], the problem of unsteady heating of a cylinder may also be solved. Omitting all the preliminary calculations, the final formulas are

$$T_1(r, t) = hT_c^4\sqrt{3at}\left[1 - \frac{(1 - 9a^2h^4T_c^{12}t^2)}{2\sqrt{3at}}(R - r)\right]^2; \tag{33'}$$

$$l(t) = \frac{2\sqrt{3at}}{1 - 9a^2h^4T_c^{12}t^2}. \tag{34'}$$

For comparatively thin cylinders ($R \leq 1/hT_c^3$), the period of unsteady heating will be short. Then we may use the simplified formulas

$$T_1(r, t) = hT_c^4\sqrt{3at}\left[1 - \frac{R - r}{2\sqrt{3at}}\right]^2; \tag{35'}$$

$$l(t) = 2\sqrt{3at}; \tag{36'}$$

$$w(t) = \sqrt{3a/t}; \tag{37'}$$

$$t_0 = R^2/12a. \tag{38'}$$

Comparing (38) with (38'), we see that the period of unsteady heating of a cylinder is less by a factor of two than that for a plate of thickness 2b equal to the cylinder diameter 2R. This result confirms similar conclusions by other authors [5], and is a point in favor of the method of solution used.

Steady heating: The beginning of this period of heating is the end of the previous period, when the heat front reaches the axis of symmetry of the body. This time may be determined from (38) or (38'). We propose to call this period steady, because, although the process still depends on time, the relative character of the temperature distribution over the cross section of the heated body is already assumed to be fixed.

To evaluate the temperature distribution function over the section in this stage, we must solve (6) or (6') with boundary conditions (7) and (8), or (7') and (8'). The initial condition is discussed below.

We shall obtain an approximate solution of the problem, using the same method of averaging correction functions. As before, we write down expressions (14), (15) and (16) for the plate. We determine the integration functions from (7) and (8) and have

$$u_2(\xi, \tau) = \sqrt[4]{1 - f_2(\tau)\beta} - \frac{f_2(\tau)\beta^2}{2}\left(1 - \frac{\xi^2}{\beta^2}\right). \tag{40}$$

The expression analogous to (15), taking into account (40), leads to the differential equation

$$\frac{df_2}{f_2} + \frac{3df_2}{4\beta f_2 \sqrt[4]{(1-\beta f_2)^3}} = -\frac{3d\tau}{\beta^2}. \quad (41)$$

By simple quadrature we obtain from (41) the following transcendental expression:

$$\ln f_2(\tau)\Big|_{\tau_0}^{\tau} - \frac{3}{2\beta}\left[\operatorname{arctg}\sqrt[4]{1-\beta f_2(\tau)} + \operatorname{arcth}\sqrt{1-\beta f_2(\tau)}\right]\Big|_{\tau_0}^{\tau} = -\frac{3}{\beta^2}(\tau-\tau_0). \quad (42)$$

Noting that the temperature at the surface is

$$u_{2s} = u_2(\beta, \tau) = \sqrt[4]{1-\beta f_2(\tau)}, \quad (43)$$

we can easily put (42) in a form very close to formula (16) of [6]

$$\mathrm{Sk}(\mathrm{Fo} - \mathrm{Fo}^0) = [\varphi(\tau) - \varphi(\tau_0)] + 0.333\,\mathrm{Sk}\,[\psi(\tau) - \psi(\tau_0)], \quad (44)$$

$$\varphi(\tau) = \frac{1}{2}[\operatorname{arctg} u_{2s} + \operatorname{arcth} u_{2s}]; \quad (45)$$

$$\psi(\tau) = -\ln(1 - u_{1s}^4); \quad (46)$$

$$u_{2s}^0 = u_{1s}(\tau_0). \quad (47)$$

Equation (47) is required as the initial condition of the problem in the second stage of heating.

The steady relative temperature distribution function (40), taking into account (43) and (5), can easily be put in the form:

$$T_2(x, t) = T_{2s}(t) - [T_c^4 - T_{2s}^4(\tau)]\frac{\mathrm{Sk}}{2T_c^3}\left(1 - \frac{x^2}{b^2}\right). \quad (48)$$

The corresponding formula for the heating of a cylinder may be obtained similarly

$$2\mathrm{Sk}(\mathrm{Fo} - \mathrm{Fo}^0) = [\varphi(\tau) - \varphi(\tau_0) + 0.333\,\mathrm{Sk}\,[\psi(\tau) - \psi(\tau_0)]; \quad (44')$$

$$T_2(r, t) = T_{2s}(t) - [T_c^4 - T_{2s}^4(t)]\frac{\mathrm{Sk}}{2T_c^3}\left(1 - \frac{r^2}{R^2}\right). \quad (48')$$

Relations (44) and (44') differ from (16) of [6] only in that in [6] the coefficient of Sk $\Psi(\tau)$ is not 0.333 but m_0/n_0, the values of which are given by a number of graphs. All the graphs show, however, that in practice m_0/n_0 differs little from 0.333, especially for large Stark numbers, i.e., for thick bodies, while for thin bodies the Stark number itself is small, so that the second term in (44) or (44') may be neglected.

Thus, the method of averaging correction functions, even in the first approximation, leads to sufficiently accurate results for practical purposes. The temperature behavior of the plate under steady radiant heating is obtained here in the form of the quadratic parabola (48) and (48').

A parabolic temperature law has previously been assumed in many papers dealing with steady radiative heating [1, 2, 5].

It is pointed out in [6], however, that "although the parabolic law allows sufficiently accurate calculation of certain data relating to the heating of a thick body, it is nevertheless unreliable and can be a source of large errors when widely used."

This warning was based on the following arguments of the authors [6]. They present the expression for rate of heating in terms of the parabolic distribution law

$$\frac{dT}{dt} = \frac{D(t)}{(x/b)^{2-n}}. \quad (49)$$

If x is put equal to zero in (49), the rate of heating on the axis of symmetry becomes infinite, which cannot be the case. Hence the authors arrived at the above conclusion.

It is quite evident, however, that if the exponent n of the parabola ≥ 2, the rate of heating will be finite everywhere. However, n must not be greater than 2, since otherwise the rate of heating on the axis of symmetry becomes zero, which is again impossible.

Therefore, if the temperature distribution function is assumed to be parabolic, it can only be of second degree, as this paper confirms analytically. Improvement in accuracy of the results obtained should be sought, not through choice of the degree of the parabola, but by finding new distribution functions differing from the customary parabolic law. This can evidently be done by solving the problem by the method of [3, 4] in the second approximation.

NOTATION

$T(x, t)$, $T(r, t)$ – respectively, temperature of plate and cylinder at time t and distance x or r from axis of symmetry; T_c = const. – temperature of heating medium; $T_s(t)$ – surface temperature; $T_0 = 0$ – initial temperature; a – thermal diffusivity; h – heat transfer coefficient; σ_v – apparent radiation coefficient; λ – thermal conductivity; 2b – plate thickness; R – cylinder radius; $l(t)$ – depth of zone of penetration of temperature perturbation inside body; $\omega(t)$ – rate of penetration of temperature perturbation; t_0 – period of unsteady heating; Sk – Stark number; Fo – Fourier number.

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